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A self-similar axisymmetric pulson in rotating stratified fluid

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Self-similar analytical nonlinear solutions to the hydrostatic Boussinesq equations are derived which describe unbalanced inertial pulsations of anticyclonic lens-like circular vortices in stably stratified rotating fluid. Any steady axisymmetric solution for a finite-volume anticyclonic vortex in the reduced-gravity approximation is shown to correspond to a set of time-periodic solutions with the amplitude of pulsations being within a range limited by the intensity of the stationary vortex. These solutions represent an extension of previous reduced-gravity analytical pulson solutions of particular forms with spatially uniform divergence of horizontal velocity oscillating in time within the vortex volume. In the self-similar form the pulson solution describes the expansion and contraction of a vortex which maintains the same spatial structure in the Lagrangian coordinates.

1. Introduction

Hydrostatic, stratified Boussinesq primitive equations (PE) are widely used for modelling large- and mesoscale variability in planetary atmospheres and oceans. However, exact non-stationary solutions for the PE are limited. Some analytic solutions have been found for finite-area lens-like vortices in the reduced-gravity shallow water formulation. One family of so-called rodon and pulson solutions is described by a set of ordinary differential equations when velocities are assumed to be linear functions of the horizontal coordinates, so that both horizontal divergence and vorticity are spatially uniform within the vortex area (see, e.g., Ball 1963; Thacker 1981; Cushman-Roisin, Heil & Nof 1985; Young 1986; Cushman-Roisin 1987; Rogers 1989; Zharnitskiy 1992). This class of exact solutions describes rotation and pulsations of elliptical anticyclonic eddies with maximum velocity at the vortex boundary; it has been generalized to the non-hydrostatic, stratified Boussinesq equations by Maas & Zahariev (1996).

Circular vortices with more realistic horizontal and vertical structure are also able to support nonlinear pulsations with inertial frequency, as described analytically for the shallow-water model by Rubino, Brandt & Hessner (1998) and recently for multi-layer reduced-gravity models by Rubino & Dotsenko (2006). In this second family of analytical non-stationary solutions the horizontal divergence of velocity oscillates in time, being spatially uniform within the vortex boundaries while the vorticity may depend on time and coordinates. The set of exact nonlinear non-stationary pulson solutions for lens-like circular vortices indicates that it might have a self-similar nature so that more general solutions could exist for even continuously stratified PE. Such self-similar solutions are described in this paper. The rest of the paper is organized as follows. In § 2 we formulate the set of PE for axisymmetric flows in rotating stratified fluid and introduce the self-similar coordinates normalized by the horizontal area of the flow variable in time. In §3 we discuss a special case of inertially pulsating vortices when the solution remains self-similar and make comparisons with previous studies. Section 4 provides a summary and conclusions.

2. Model development

2.1. *Axisymmetric formulation*

We consider a stratified, Boussinesq fluid on a rotating plane. Assuming axisymmetry, we write the governing equations for an inviscid flow with velocity (u, v, w) in the cylindrical coordinates (r, θ, z):

$$\frac{\mathrm{D}u}{\mathrm{D}t} + \frac{\partial \phi}{\partial r} = \frac{v^2}{r} + fv \equiv \frac{m^2}{r^3} - \frac{f^2 r}{4}, \tag{1}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + u\left(\frac{v}{r} + f\right) = 0, \qquad \text{or} \qquad \frac{\mathrm{D}m}{\mathrm{D}t} = 0, \tag{2}$$

$$\frac{\partial ru}{\partial r} + r\frac{\partial w}{\partial z} = 0, \tag{3}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial \phi}{\partial z} = 0, \tag{4}$$

where f is the Coriolis parameter, $m \equiv vr + fr^2/2$ is the absolute angular momentum which is conserved by fluid parcels as is the buoyancy, $g(\rho - \rho_0)/\rho_0 = -\partial\phi/\partial z$, related to the geopotential, ϕ, by the hydrostatic approximation, g is the acceleration due to gravity, ρ is the density, ρ_0 is its reference value, and

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}. \tag{5}$$

2.2. *Coordinate transformation*

We consider a flow between horizontal level $z = 0$ and an isopycnal (i.e. constant buoyancy) surface inside some volume which may depend on time, t, and seek the solution for the Lagrangian variables in the form

$$m = M(R, Z), \qquad \frac{\partial \phi}{\partial z} = -B(R, Z), \qquad (R, Z) \equiv \left(\frac{r}{\sqrt{S}}, Sz\right), \tag{6}$$

where $S(t)$ may depend on time. In this case, from (2)–(3) we see that radial and vertical velocities depend linearly on spatial coordinates

$$(u, w) = \left(\frac{\dot{S}r}{2S}, -\frac{\dot{S}z}{S}\right), \tag{7}$$

so that the condition of zero horizontal velocity it satisfied at the vortex center $r = 0$ and the condition of zero vertical velocity is imposed at the level $z = 0$ corresponding to the geopotential maximum. The kinematic condition at the isopycnal vortex boundary is satisfied because the fluid parcels do not penetrate isopycnal surfaces. Note, the horizontal divergence is spatially uniform: $\nabla \cdot \boldsymbol{v} = \dot{S}/S$ within the flow volume. The hydrostatic relation is satisfied if

$$\phi = \frac{\Phi(R, Z)}{S}, \qquad \frac{\partial \Phi}{\partial Z} = -B, \tag{8}$$

Finally, (1) provides a relation between M and Φ in the form

$$\frac{M^2}{R^4} - \frac{1}{R}\frac{\partial \Phi}{\partial R} = \frac{1}{4}(f^2 S^2 - \dot{S}^2 + 2S\ddot{S}) \equiv \frac{f^2}{4}(1-a^2), \tag{9}$$

where the right-hand side is denoted by $f^2(1-a^2)/4$ for convenience. The self-similar solution exists only if the right-hand side of (9) does not depend on time because the left-hand side does not depend on time, and therefore a must be a constant. This equation allows the absolute momentum (and azimuthal velocity) to be calculated for a given Φ (and vice versa):

$$M = R\sqrt{\frac{f^2R^2}{4}(1-a^2) + R\frac{\partial \Phi}{\partial R}}, \qquad v = -\frac{fr}{2} + \frac{M}{r}. \tag{10}$$

Here we see that M remains real when the value of a is limited by

$$a^2 \leqslant 1 - A_m, \qquad A_m = \frac{4}{f_0^2}\max\left(-\frac{\partial \Phi}{R\partial R}\right), \tag{11}$$

where A_m characterizes the non-dimensional vortex intensity.

It is well known that for a stationary solution, if $S=1, a=0$, equation (9) describes an axisymmetric vortex in the gradient wind balance (e.g. McWilliams 1985). Note that rotation in the stationary vortex could be either cyclonic ($v>0$) or anticyclonic ($v<0$). In particular, the spatial structure of observed oceanic eddies has been related to vortices with a finite core of potential vorticity anomaly by Sutyrin (1989). Although $(u, w)=(0,0)$ in the stationary vortex, this gradient balance state can be also used to describe slow balanced evolution of a circular vortex due to frictional effects which generate weak radial–vertical circulation (e.g. Sutyrin 1992).

3. Analytic self-similar pulson solutions

One can see that (9) is also satisfied if S oscillates with the inertial period

$$S = 1 + a\sin(ft), \tag{12}$$

where $0<a\leqslant\sqrt{1-A_m}<1$ according to (11), so that the physically realistic demand that $S>0$ is satisfied. For this new set of nonlinear non-stationary solutions depending on a, the spatial distribution $\phi(R,Z)$ is the same as for the stationary solution except that its amplitude pulsates inversely proportionally to S in order to provide the hydrostatic balance described by (8). Therefore, it has physical meaning only for finite-volume vortices if $\Phi=0$ for $R>R_B(Z)$. The actual vortex radius pulsates with time as $r_B(t,z)=\sqrt{S}R_B(zS)$. Correspondingly, the isopycnal boundary for $B(R_B(Z),Z)\equiv g'$ (the reduced gravity) becomes deeper or shallower following oscillations in S: $z_B = R_B^{-1}(rS^{-1/2})/S$. It outcrops at the level $z=0$ at variable radial distance $r_0 = R_B(0)\sqrt{1+a\sin(ft)}$. Therefore, such a solution can describe an anticyclonic (warm-core) lens-like vortex with all isopycnals outcropping at $R_0(B)\leqslant R_B(0)$ at the level $z=0$ (figure 1). In particular, all isopycnals may outcrop at the same radius $R_B(0)$. Note that the vertical velocity described by (7) increases from zero at the level $z=0$ to a maximum at the isopycnal vortex boundary while Φ decreases from a maximum at the level $z=0$ to zero at the vortex boundary overlying deep motionless fluid, as often assumed in the reduced-gravity approximation.

Correspondingly, the azimuthal velocity calculated from (10) for $a>0$ deviates from the stationary gradient balance to compensate for the impact of pulsating radial velocity. Thus, such an unbalanced solution has non-zero agradient velocity (cf. Sutyrin

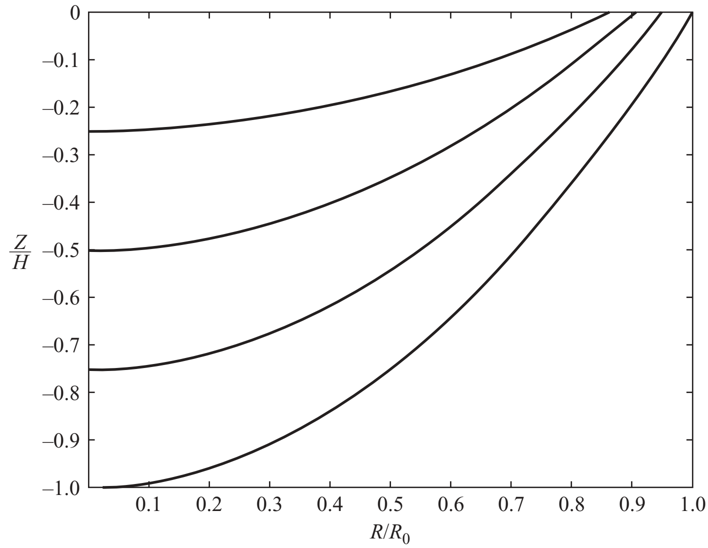

FIGURE 1. Characteristic contours of buoyancy for B linearly proportional to Z at the vortex centre $B(0, Z) = -g'Z/H$; H is the lens thickness: $R_B(-H) = 0$.

2004) and remains unbalanced because inertia–gravity waves are trapped inside the edge of such lens-like vortices: they are not able to propagate through outcropping isopycnals.

In particular, if we consider $\Phi(R, 0)$ at the reference level with zero horizontal gradient at the vortex edge,

$$\frac{\partial \Phi}{\partial R} = 0 \qquad \text{at} \qquad R = R_0, \tag{13}$$

then the azimuthal velocity vanishes at the vortex edge $r = r_0$ only for a stationary state; while for $a > 0$ the edge velocity oscillates:

$$v_0(t) \equiv v(t, r_0) = \frac{f R_0}{2}\left(\sqrt{\frac{1-a^2}{S}} - \sqrt{S}\right), \tag{14}$$

in order to provide conservation of absolute angular momentum $M(R_0, 0)$. At the maximum expansion when $S = 1 + a$, the edge velocity becomes anticyclonic:

$$v_0 = -v_a, \qquad v_a = \frac{af R_0}{\sqrt{1+a} + \sqrt{1-a}}, \tag{15}$$

while at the maximum contraction when $S = 1 - a$, the edge velocity becomes cyclonic: $v_0 = v_a$. At the intermediate times when $S = 1$ we see that anticyclonic rotation is stronger at all radii when a is larger, as shown in figure 2 for

$$\Phi(R, 0) = \Phi_m\left[1 + \cos\left(\frac{\pi R}{R_0}\right)\right]. \tag{16}$$

In this case $A_m = 4\pi^2 \Phi_m / f_0^2 R_0^2$ and the absolute vorticity remains positive when $a < \sqrt{1 - A_m}$.

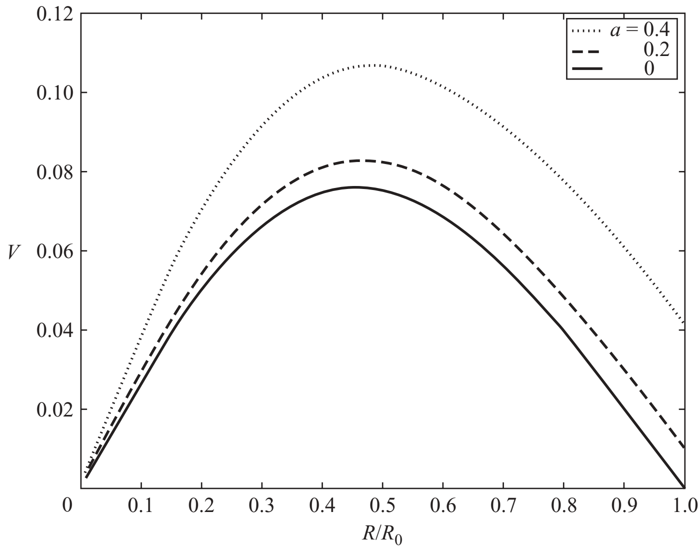

FIGURE 2. Vortex velocity $V(R) = -v/fR_0$ for different a calculated from (10) and (16) for $\Phi_m = f^2R_0^2/5\pi^2$.

The radial velocity in all previously found axisymmetric pulson solutions corresponds to that given by (7) and (12):

$$u = \frac{rf}{2}\frac{a\cos(ft)}{1 + a\sin(ft)}, \tag{17}$$

while the difference was in the particular choice of Φ: in the shallow-water models a paraboloidal shape $\Phi(R) = \Phi_m(1 - R^2/R_0^2)$ with maximum azimuthal velocity at the lens edge was considered by Young (1986), Cushman-Roisin (1987), Rogers (1989), Zharnitskiy (1992), while more general polynomial representations were suggested by Rubino *et al.* (1998). Generalizations to the multi-layer model with the same radial velocity (17) were obtained by Rubino & Dotsenko (2006).

4. Conclusions

These self-similar solutions demonstrate that during the inertial period the structure of axisymmetric pulsons remains essentially the same in properly chosen Lagrangian coordinates. A simple analytic expression for non-stationary PE solutions is found for a fairly arbitrary horizontal and vertical vortex structure corresponding to a stationary lens-like anticyclone. It depends on the amplitude, a, of the vortex area pulsations which is limited depending on the vortex intensity according to (11). These exact solutions can be used for assessing laboratory and numerical models with layer outcropping (cf. Sun, Bleck & Chassignet 1993; Rubino, Hessner & Brandt 2002; Rubino & Brandt 2003).

In a geophysical context, both the radial and vertical structure of such solutions for lens-like stratified anticyclones can be chosen quite realistically (see, e.g., figure 1) to be applied to a variety of surface-intensified rings with the level $z = 0$ defined by the maximum azimuthal velocity near the ocean surface (e.g. Olson 1991) as well as to the abundant intrathermocline coherent anticyclones with the level $z = 0$ at the depth where their azimuthal velocity has its maximum (e.g. McWilliams 1985; Richardson, Bower & Zenk 2000).

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